Folien zur Vorlesung am 08.04.2025

3D Computer Vision

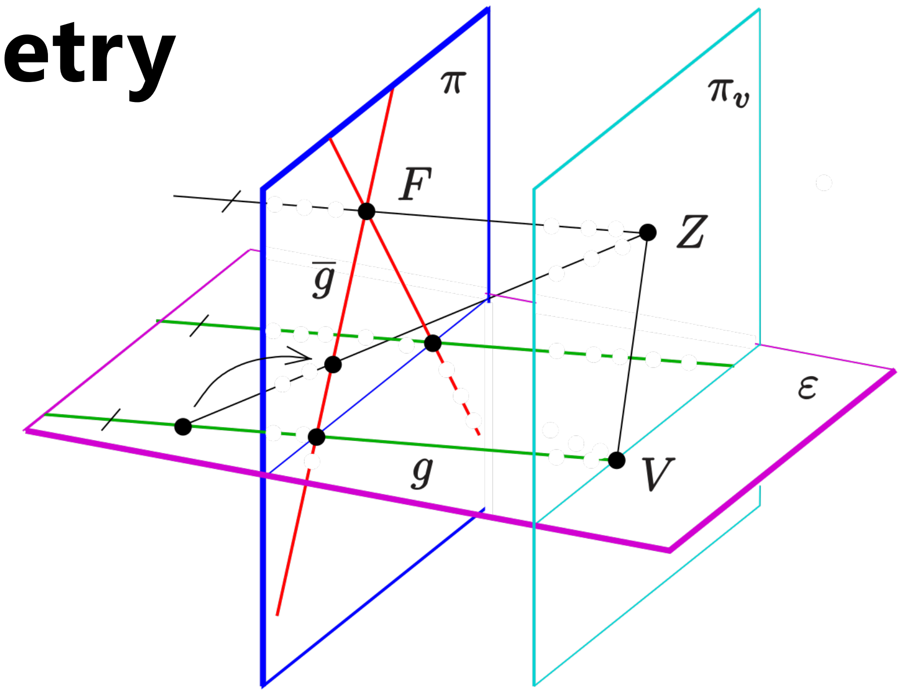
PROJEKTIVE GEOMETRIE

Geometry

On the one hand, **geometry** refers to 2D+3D Euclidean geometry, which is also taught in mathematics lessons and which deals with points, straight lines, planes, distances, angles, etc.

On the other hand, the term geometry encompasses a number of large areas of mathematics whose relationship to elementary geometry is *difficult for the layperson* to recognize. This applies in particular to the **modern concept of geometry**, which generally refers to the study of invariant quantities.

Projective Geometry

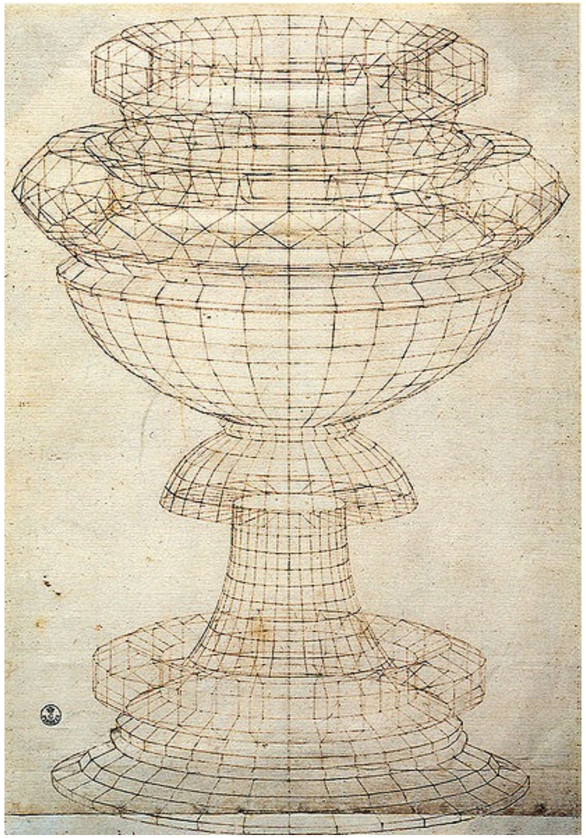


Readings

- Mundy and Zisserman. *Geometric Invariance in Computer Vision*, Appendix: Projective Geometry for Machine Vision, MIT Press, 1992, **(read 23.1-23.5, 23.10)**
  - available online: http://www.cs.cmu.edu/~ph/869/papers/zisser-mundy.pdf
- Stan Birchfield. An Introduction to Projective Geometry (for computer vision)
  - available online: https://robotics.stanford.edu/~birch/projective/projective.pdf

Projective geometry—what's it good for?

- Uses of projective geometry
  - Drawing
  - Measurements
  - Mathematics for projection
  - Undistorting images
  - Camera pose estimation
  - **Object recognition**



Paolo Uccello

Applications of projective geometry

Vermeer's *Music Lesson*

Reconstructions by Criminisi et al.

Making measurements in images

WARBY PARKER

Measure your pupillary distance (PD)

Your PD is the distance between your pupils. To measure it, follow the instructions below — once you submit your photo, our team of experts will determine your PD and email you once we've applied it to your order.

1

**Wearing glasses?
Take 'em off before you get started.**

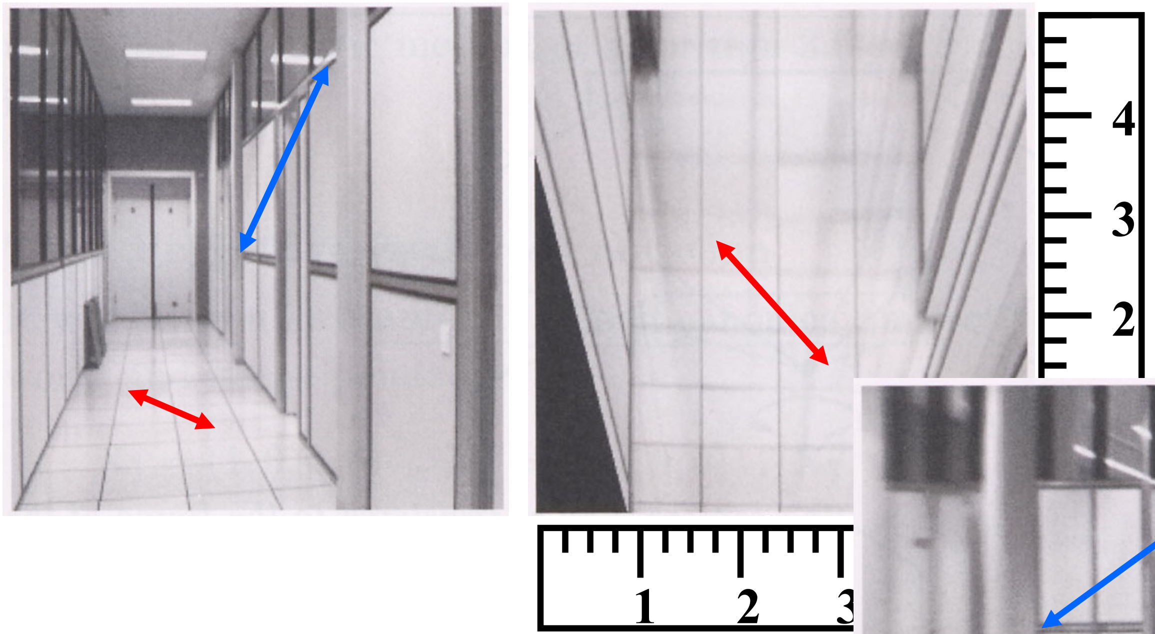
2

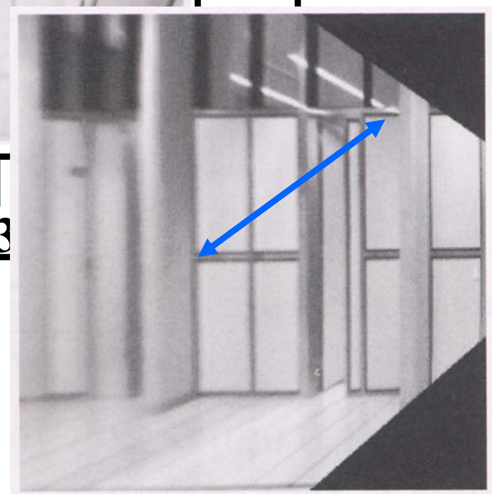
Hold up any card with a magnetic strip (we use this for scale).

3

Look straight ahead and snap a photo.

Measurements on planes

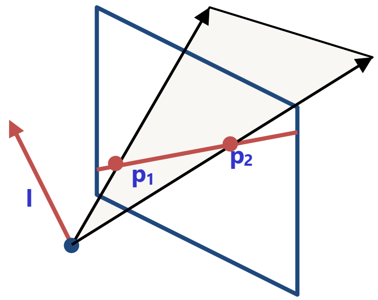




Approach: unwarp then measure

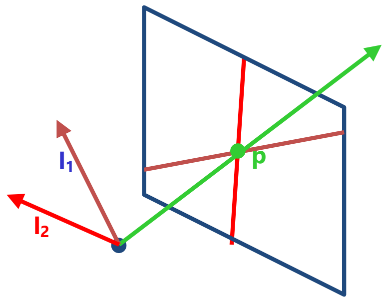
Point and line duality

- A line $\mathbf{l}$ is a homogeneous 3-vector
- It is $\perp$ to every point (ray) $\mathbf{p}$ on the line: $\mathbf{l} \cdot \mathbf{p} = 0$



What is the line $\mathbf{l}$ spanned by points $\mathbf{p_1}$ and $\mathbf{p_2}$?

- $\mathbf{l}$ is $\perp$ to $\mathbf{p_1}$ and $\mathbf{p_2}$ $\Rightarrow$ $\mathbf{l} = \mathbf{p_1} \times \mathbf{p_2}$
- $\mathbf{l}$ can be interpreted as a *plane normal*

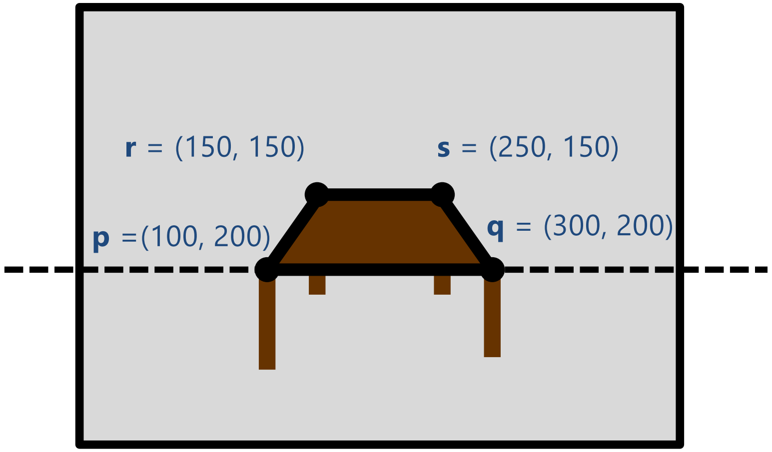


What is the intersection of two lines $\mathbf{l_1}$ and $\mathbf{l_2}$?

- $\mathbf{p}$ is $\perp$ to $\mathbf{l_1}$ and $\mathbf{l_2}$ $\Rightarrow$ $\mathbf{p} = \mathbf{l_1} \times \mathbf{l_2}$

Points and lines are *dual* in projective space

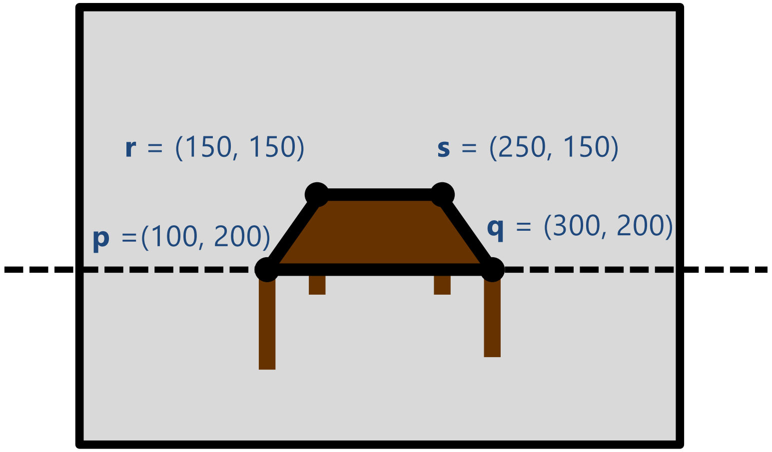
Example



What is the line passing through points **p** and **q**?

$$\mathbf{p} \times \mathbf{q} = \begin{bmatrix} 100 \\ 200 \\ 1 \end{bmatrix} \times \begin{bmatrix} 300 \\ 200 \\ 1 \end{bmatrix} = \begin{bmatrix} 200 \cdot 1 - 200 \cdot 1 \\ 300 \cdot 1 - 100 \cdot 1 \\ 100 \cdot 200 - 300 \cdot 200 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \\ -40000 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ -200 \end{bmatrix}$$

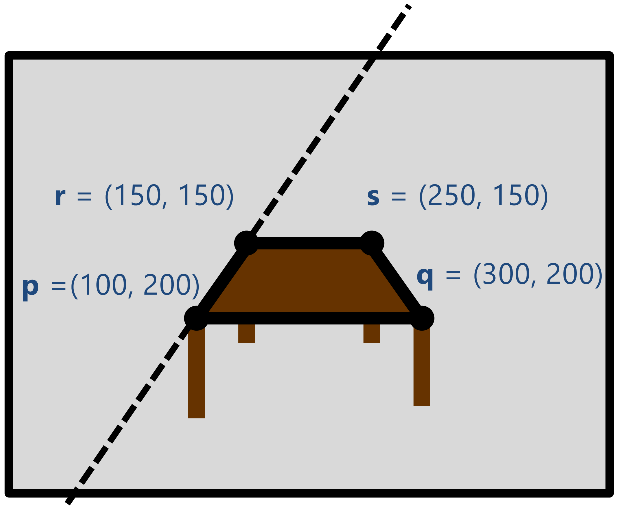
Example



How do we interpret the line $\ell = \begin{bmatrix} 0 \\ 1 \\ -200 \end{bmatrix}$?

Answer, the set of points (*x*, *y*) such that $\ell \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$, i.e., $y - 200 = 0$

Example

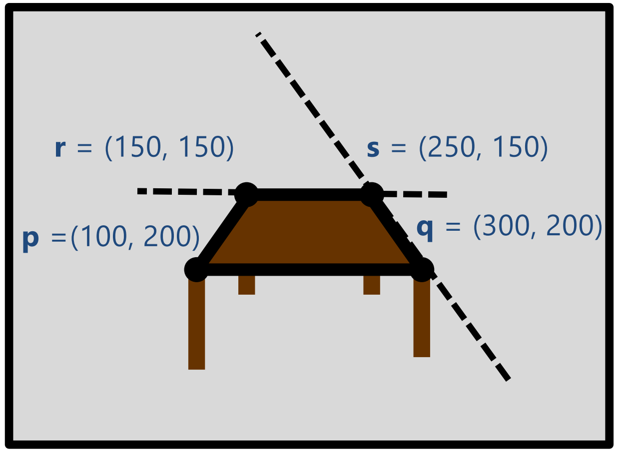


What is the line passing through points **p** and **r**?

$$\mathbf{p} \times \mathbf{r} = \begin{bmatrix} 100 \\ 200 \\ 1 \end{bmatrix} \times \begin{bmatrix} 150 \\ 150 \\ 1 \end{bmatrix} = \begin{bmatrix} 200 \cdot 1 - 150 \cdot 1 \\ 150 \cdot 1 - 100 \cdot 1 \\ 100 \cdot 150 - 150 \cdot 200 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ -15000 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ -300 \end{bmatrix}$$

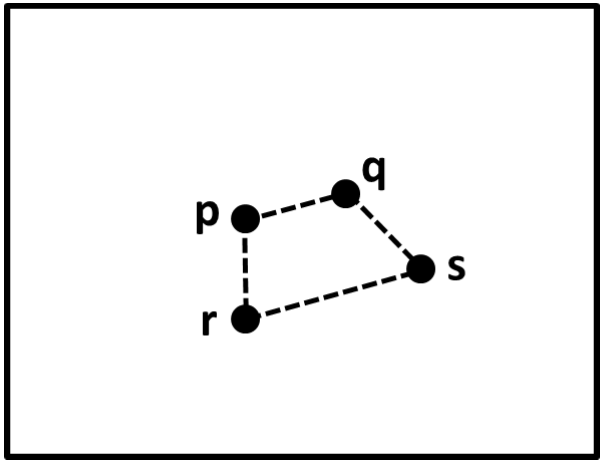
i.e., all points (x, y) such that $x + y = 300$

Your turn



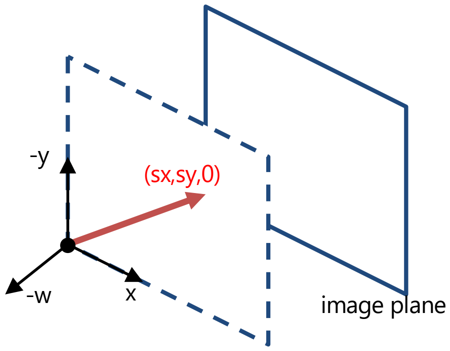
What are the lines passing through points **r** and **s** resp. **s** and **q**?

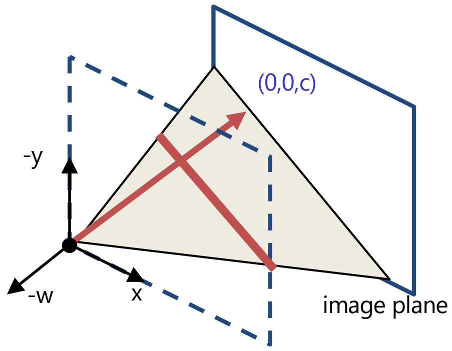
Question time



Consider the above image, with four points **p**, **q**, **r**, **s**, labeled (assume these are 2D homogeneous points). What is a simple expression for the point of intersection between the lines **pr** and **qs**?

Ideal points and lines





- Ideal point ("point at infinity")
  - p ≅ (sx, sy, 0) – parallel to image plane
  - It has infinite image coordinates
- Ideal line ("line at infinity")
  - l ≅ (0, 0, c) – parallel to image plane
  - Corresponds to a line in the image (finite coordinates)
    - goes through image origin (*principal point*)

2D projective geometry

Point	$\tilde{P} = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$	Line	$L = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
Ideal point	$\tilde{P}_{ideal} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$	Ideal Line	$L_{ideal} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$
Connect 2 points	$L = \tilde{P}_1 \times \tilde{P}_2$	Intersect 2 lines	$P = L_1 \times L_2$
Colinearity	$det[\tilde{P}_1, \tilde{P}_2, \tilde{P}_3] = 0$	intersect 3 lines	$det[L_1, L_2, L_3] = 0$
incidence point on line	$\tilde{P}^T L = 0$	Line through a point	$L^T \tilde{P} = 0$

[Source: Keivan Kiyanfar Blog entry about Projective Geometry]

3D projective geometry

Point	$\tilde{P} = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$	Plane	$\pi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$
Ideal point	$\tilde{P} = \begin{pmatrix} X \\ Y \\ Z \\ 0 \end{pmatrix}$	Ideal Plane	$\pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}$
3 points define a plane	$\begin{pmatrix} \tilde{P}_1^T \\ \tilde{P}_2^T \\ \tilde{P}_3^T \end{pmatrix} \pi = 0$	Intersection of 3 plane	$\begin{pmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{pmatrix} \tilde{P} = 0$
Coplanarity	$det[\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4] = 0$	Intersection of 4 planes	$det[\pi_1, \pi_2, \pi_3, \pi_4] = 0$
Incidence of point & plane	$\tilde{P}^T \pi = 0$	plane through a point	$\pi^T \tilde{P} = 0$

[Source: Keivan Kiyanfar Blog entry about Projective Geometry]